Exercise 62

(a) Sketch the graph of the function $g(x) = x + |x|$.

(b) For what values of x is g differentiable?

(c) Find a formula for g'.

Solution

Use the definition of the derivative to find g'.

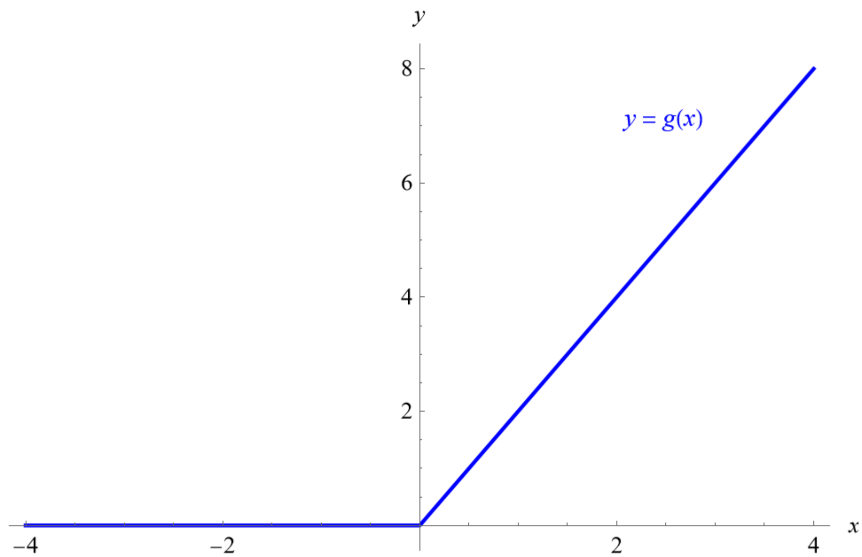
$$
\begin{aligned}
g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} &= \lim_{h\to 0} \frac{[(x+h) + |x+h|] - (x + |x|)}{h} \\
&= \lim_{h\to 0} \frac{x + h + |x+h| - x - |x|}{h} \\
&= \lim_{h\to 0} \frac{h + |x+h| - |x|}{h} \\
&= \lim_{h\to 0} \left(\frac{h}{h} + \frac{|x+h| - |x|}{h}\right) \\
&= 1 + \lim_{h\to 0} \frac{|x+h| - |x|}{h} \\
&= 1 + \lim_{h\to 0} \frac{\sqrt{(x+h)^2} - \sqrt{x^2}}{h} \\
&= 1 + \lim_{h\to 0} \frac{\sqrt{(x+h)^2} - \sqrt{x^2}}{h} \cdot \frac{\sqrt{(x+h)^2} + \sqrt{x^2}}{\sqrt{(x+h)^2} + \sqrt{x^2}} \\
&= 1 + \lim_{h\to 0} \frac{(x+h)^2 - (x^2)}{h\left[\sqrt{(x+h)^2} + \sqrt{x^2}\right]} \\
&= 1 + \lim_{h\to 0} \frac{(x^2 + 2xh + h^2) - x^2}{h\left[\sqrt{(x+h)^2} + \sqrt{x^2}\right]} \\
&= 1 + \lim_{h\to 0} \frac{2xh + h^2}{h\left[\sqrt{(x+h)^2} + \sqrt{x^2}\right]} \\
&= 1 + \lim_{h\to 0} \frac{2x + h}{\sqrt{(x+h)^2} + \sqrt{x^2}} \\
&= 1 + \frac{2x}{\sqrt{(x)^2} + \sqrt{x^2}} \\
&= 1 + \frac{2x}{2\sqrt{x^2}} = 1 + \frac{x}{\sqrt{x^2}} = 1 + \frac{x}{|x|} = 1 + \operatorname{sgn} x
\end{aligned}
$$

$g(x) = x + |x|$ is not differentiable at 0 because there's $|x|$ in the denominator, and for any rational function the denominator cannot be zero.

$$|x| \neq 0$$

$$x \neq 0$$

The domain of $g'(x)$ is $\{x \mid x \neq 0\}$. Below is a graph of $g(x)$ versus x.



Below is a graph of $g'(x)$ versus x.